Fundamental theorem of Calculus, Green's theorem, and the Poincaré Lemma

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The two forms of the Fundamental Theorem of Calculus:

(α) Real valued cts functions f defined on the interval $[a,b] \subseteq \mathbb{R}$ admit a primitive, i.e.,

$\exists$ $F : [a,b] \to \mathbb{R}$ such that

$$F' = f \text{ on } (a,b).$$

(β) The integral is an anti-derivative: If g is a cts function on $[a,b]$ that is diff on (a,b), and if g' is cts on $[a,b]$, then

$$\int_a^b g' = g(b) - g(a).$$

How about generalizations to multi-variable functions ??

Let $f : U \longrightarrow \mathbb{R}^2$, $U \subseteq \mathbb{R}^2$, open be a smooth function.

Ask: If there exist a smooth function $F : U \longrightarrow \mathbb{R}$ such that

$$\frac{\partial F}{\partial x_1} = f_1 \text{ and } \frac{\partial F}{\partial x_2} = f_2 \, ?$$

If such a function exists, then we must have

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial f_2}{\partial x_1} .$$

Therefore, a necessary condition for the existence of a primitive is

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}.$$

A natural question is to ask if this obvious necessary condition is also sufficient for the existence of the primitive.

What is your guess?

The surprise answer — depends on the geometry of the open set U.

Thus, let $U = \mathbb{R}^2 \setminus \{(0,0)\}$ and $f : U \to \mathbb{R}^2$ be the function given by the formula:

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

We have

$$\frac{\partial f_2}{\partial x_1} = \frac{x_1^2 + x_2^2 - 2x_1^2}{(x_1^2 + x_2^2)^2}$$

and

$$\frac{\partial f_1}{\partial x_2} = \frac{-(x_1^2 + x_2^2) + 2x_2^2}{(x_1^2 + x_2^2)^2}.$$

Therefore $\frac{\partial f_2}{\partial x_1} = \frac{\partial f_1}{\partial x_2}$ and the necessary condition is met.

However, we claim that the function f has no primitive, that is, there is no function $F: U \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial x_1} = f_1 \text{ and } \frac{\partial F}{\partial x_2} = f_1 .$$

Assume to the contrary, namely, that such a function exists.

Then
$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta$$
$$= F(1,0) - F(1,0) = 0.$$

On the other hand, using the chain rule, we also have
$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = 2\pi$$
leading to a contradiction.

To prove this, first note following.

(i) $f_1(\cos\theta, \sin\theta) = -\frac{x_2}{x_1^2+x_2^2} = -\sin\theta$

(ii) $f_2(\cos\theta, \sin\theta) = \frac{x_1}{x_1^2+x_2^2} = \cos\theta$

and

(*) $$\frac{\partial f_2}{\partial x_1} = \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2} = \frac{\partial f_1}{\partial x_1}.$$

Recall the chain rule: Suppose $\Theta : \mathbb{R} \to \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}$. Then

$$D(F\circ\Theta)(t) = DF(\Theta(t))\,D\Theta(t).$$

Thus we have

$$D(F\circ(\cos,\sin))(t) = DF(\cos t, \sin t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$= \left(\frac{\partial F}{\partial x_1}(\cos t, \sin t), \frac{\partial F}{\partial x_2}(\cos t, \sin t)\right)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

We therefore see that

$$\left(\frac{d}{dt}F\right)(\cos t, \sin t)$$

$$= -\sin t \frac{\partial F}{\partial x_1}(\cos t, \sin t) + \cos t \frac{\partial F}{\partial x_2}(\cos t, \sin t)$$

$$= -\sin t \; f_1(\cos t, \sin t) + \cos t \; f_2(\cos t, \sin t)$$

$$= \sin^2 t + \cos^2 t = 1.$$

Finally, we therefore find that

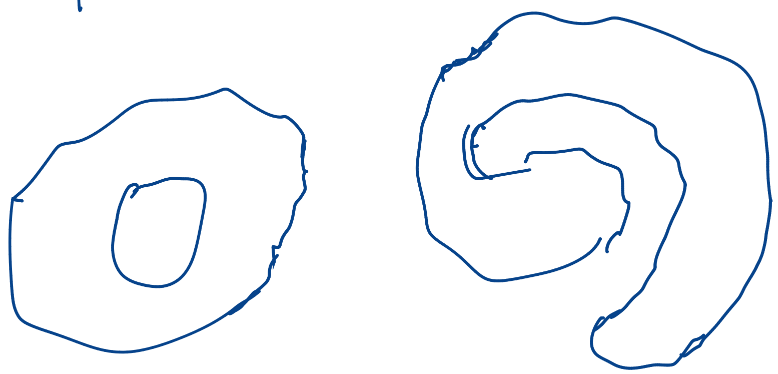
$$\int_0^{2\pi} \left(\frac{d}{dt}F\right)(\cos t, \sin t)\, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi.$$

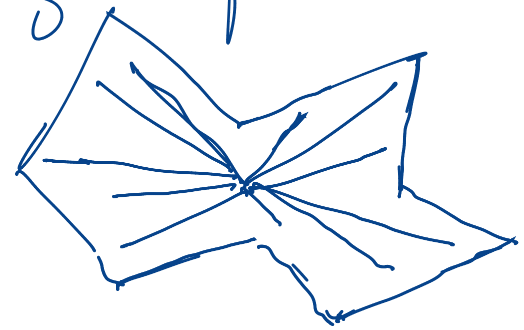
Poincare Lemma for star shaped domains.

<u>Definition</u>: A subset X of $\mathbb{R}^2$ is said to be star shaped if $\exists$ a point $x_0 \in X$ such that the line segment $\{tx_0 + (1-t)x : t \in [0,1]\}$ is in X for all $x \in X$.

not star shaped



Star-shaped.



The following theorem gives a sufficient condition for the existence of a primitive.

Theorem: Let $U \subseteq \mathbb{R}^2$ be a star shaped open set. For any smooth function $f: U \to \mathbb{R}^2$ that meets the necessary condition, there exists a primitive.

Proof: Assume that $x_0 = 0 \in X$ and X is star shaped wrt 0. Consider the function $F: X \to \mathbb{R}$ given by the formula

$$F(x_1, x_2) = \int_0^1 x_1 f_1(tx_1, tx_2) + x_2 f_2(tx_1, tx_2)\, dt$$

Then we have

$$\frac{\partial F}{\partial x_1}(x_1, x_2) = \int_0^1 \Big\{ f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2) \Big\}\, dt.$$

Now observe that

$$\frac{d}{dt}\left(t f_1(tx_1, tx_2)\right) = f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_1}{\partial x_2}(tx_1, tx_2)$$

Substituting this, we have

$$\frac{\partial F}{\partial x_1}(x_1, x_2) = \int_0^1 \left\{ \frac{d}{dt}\left(t f_1(tx_1, tx_2)\right) + tx_2 \left(\frac{\partial f_2}{\partial x_1}(tx_1, tx_2) - \frac{\partial f_1}{\partial x_2}(tx_1, tx_2)\right\} dt$$

By assumption, $\frac{\partial f_2}{\partial x_1}(tx_1, tx_2) = \frac{\partial f_1}{\partial x_2}(tx_1, tx_2)$, therefore, it follows that

$$\frac{\partial F}{\partial x_1}(x_1, x_2) = \int_0^1 \frac{d}{dt}\left(t f_1(tx_1, tx_2)\right) dt = t f_1(tx_1, tx_2)\Big|_0^1 = f_1(x_1, x_2).$$

Similarly, we verify that $\frac{\partial F}{\partial x_2}(x_1, x_2) = f_2(x_1, x_2)$ completing the proof. $\blacksquare$

What about the other half of the Fundamental theorem of calculus?

Double integrals and Path integrals:

Let $U \subseteq [a,b] \times [c,d]$ be a connected open subset of $\mathbb{R}^2$. Let (x_i, x_{i+1}) and (y_j, y_{j+1}) be partitions of $[a,b]$ and $[c,d]$ respectively such that $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \subseteq U$.

The limit of the Riemann sum

$$\sum \sum f(x_i, y_j) \Delta x_i \Delta y_j , \quad f : U \to \mathbb{R},$$

where $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_j = y_{j+1} - y_j$, as Δx_i and Δy_j approach 0 is defined to be the double integral $\iint_U f \, dx \, dy$.

Assume that ∂U is a smooth curve, i.e., there is a smooth map $[0,1] \xrightarrow{\gamma} \mathbb{R}^2$, $\gamma'(t) > 0$, such that $\partial U = \{ \gamma(t) : t \in [0,1] \}$. The line integral $\int_{\partial U} F$ is defined to be the Riemann integral $\int_0^1 F(t) \cdot \gamma'(t) \, dt$.

Green's Theorem:

$$\iint_U \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial U} F.$$

Back to Primitives.

Find invariants for U to detect if primitives must exist for all smooth vector fields $F: U \to \mathbb{R}^2$, which satisfy the obvious necessary condition.

Let $C^\infty(U, \mathbb{R}^k)$ be the space of smooth functions $f: U \longrightarrow \mathbb{R}^k$. It is a linear space.

Define $\text{grad}: C^\infty(U,\mathbb{R}) \longrightarrow C^\infty(U,\mathbb{R}^2)$ and

$\text{rot}: C^\infty(U,\mathbb{R}^2) \longrightarrow C^\infty(U,\mathbb{R})$

$$\text{grad}(\varphi) = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}\right), \quad \varphi \in C^\infty(U,\mathbb{R})$$

and

$$\text{rot}(\varphi) = \frac{\partial \varphi_1}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_1}, \quad \varphi = (\varphi_1, \varphi_2) \in C^\infty(U,\mathbb{R}^2).$$

Observe that $\text{rot} \circ \text{grad} = \{0\}$, which amounts to saying that $\text{Kernel rot} \supseteq \text{Im grad}$.

Both grad and rot are linear maps, therefore $\text{Im}(\text{grad})$ is a subspace of $\text{Ker}(\text{rot})$.

These spaces are infinite dimensional.

However, $\dim H^1(U) = \text{Ker}(\text{rot}) / \text{Im}(\text{grad}) < \infty$.

Theorem. The sequence $0 \to C^\infty(U,\mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U,\mathbb{R}^2) \to 0$ of vector spaces along with the maps rot, grad is a complex.

If U is star-shaped, then $H^1(U) = \{0\}$. On the other hand if $U = \mathbb{R}^2 \setminus \{(0,0)\}$, then $H^1(U) \neq \{0\}$.

Thank You!